The other initial conditions are designated by the constants α_1, α_2, α_3, and α_4. These are the initial components of the thrust acceleration vector and their initial rates and are the variables iterated on to achieve the desired final rendezvous or intercept conditions. In particular,

$$\ddot{r}_o = r_o\dot{\theta}_o^2 - \frac{\mu}{r_o^2} + \alpha_1 \tag{30a}$$

$$r_o\ddot{\theta}_o = \alpha_2 - 2\dot{r}_o\dot{\theta}_o \tag{30b}$$

along with

$$r_o^{iii} = \dot{r}_o\left(\frac{2\mu}{r_o^3} - 3\dot{\theta}_o^2\right) + \alpha_3 + 2\dot{\theta}_o\alpha_2 \tag{31a}$$

$$r_o\theta_o^{iii} = \frac{3\dot{r}_o}{r_o}(2\dot{r}_o\dot{\theta}_o - \alpha_2) + 2\left(\frac{\mu\dot{\theta}_o}{r_o^2} - r_o\dot{\theta}_o^3\right) - 2\dot{\theta}_o\alpha_1 + \alpha_4 \tag{31b}$$

These general initial conditions simplify for a circular initial orbit of radius r_c as follows:

$$r_o = r_c; \qquad \theta_o = 0 \tag{32}$$

$$\dot{r}_o = 0; \qquad \dot{\theta}_o = \left(\frac{\mu}{r_c^3}\right)^{1/2} \tag{33}$$

$$\ddot{r}_o = \alpha_1; \qquad r_c\ddot{\theta}_o = \alpha_2 \tag{34}$$

and

$$r_o^{iii} = \alpha_3 + 2\left(\frac{\mu}{r_c^3}\right)^{1/2}\alpha_2 \tag{35a}$$

$$r_c\theta_o^{iii} = \alpha_4 - 2\left(\frac{\mu}{r_c^3}\right)^{1/2}\alpha_1 \tag{35b}$$

In some applications fewer than four α parameters need to be iterated. One example is a time-open, angle-open, circle-to-circle low-Earth-orbit (LEO) to geosynchronous-Earth-orbit (GEO) transfer. In this case one can select $\alpha_1 = 0$ (tangential initial thrust) and choose α_2 to be a realistic value of initial thrust magnitude divided by the initial mass. The numerical integration is terminated at a time T for which GEO altitude is reached and the values of α_3 and/or α_4 are iterated to achieve zero final eccentricity. For very low thrust levels the eccentricity remains very small for the entire transfer.

Numerical Example

For an initial thrust of 4 N, mass of 1000 kg and 15 kW power, an optimal LEO to GEO transfer is obtained by integration of Eqs. (26) and (27) from $r = 1.05$ to 6.61 Earth radii with $\alpha_1 = \alpha_3 = \alpha_4 = 0$ and $\alpha_2 = 4.08 \times 10^{-4}$ in canonical units ($\mu = 1$). The transfer time T is 1440 time units, which corresponds to 13.5 days. The optimal final thrust magnitude is 2.47 N with a propellant consumption of 383 kg. By contrast, running at (non-optimal) constant 4N thrust magnitude in the optimal thrust direction takes a shorter time equal to 10.0 days, but requires a significantly larger 463 kg of propellant. Assuming this saving of 80 kg of propellant (8% of the initial mass) can be exchanged for an equal amount of payload, using the optimal trajectory can increase a 160 kg payload to 240 kg, representing an increase of 50%.

Concluding Remarks

For a power-limited spacecraft, the equations of motion and the necessary conditions for an optimal trajectory have been combined into a single fourth-order differential equation for the position vector. Every solution to this equation represents an optimal trajectory through the specified gravitational field. The equation for an arbitrary gravitational field has been derived and specialized to the inverse-square gravitational field. An example trajectory is described and the significant increase in payload attainable by an optimal trajectory is illustrated.

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Attitude Determination in Higher Dimensions

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Introduction

GENERALLY, one has little cause to estimate an attitude in spaces of dimension higher than three. This exercise, however, will afford an insight into the workings of a well-known attitude determination algorithm in three dimensions. In addition, should the dimensionality of our world ever increase without notice, we will be all the better prepared.

An $n \times n$ proper orthogonal matrix A satisfies

$$A^TA = I_{n\times n} \tag{1}$$

$$\det A = 1 \tag{2}$$

Equation (1) is equivalent to $n(n+1)/2$ constraints on the matrix A. Hence, A can have only $n(n-1)/2$ free parameters, as remarked by Bar-Itzhack[1] and Bar-Itzhack and Markley.[2] Thus, A may be represented in terms of matrices of manifestly smaller parameter dimension. For example,

$$A = \exp\{\Theta\} \tag{3}$$

where Θ is an $n \times n$ antisymmetric matrix whose independent elements are the n-dimensional generalization of the rotation vector.[1] Likewise, one may write[2]

$$A = (I + G)(I - G)^{-1} \tag{4}$$

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where G is an $n \times n$ antisymmetric matrix, whose independent elements are the n-dimensional equivalent of the Gibbs vector.[3,4] The matrices $\mathbf{\Theta}$ and G are related by

$$G = \tanh(\mathbf{\Theta}/2) \tag{5}$$

Writing

$$G = [[g]], \qquad \mathbf{\Theta} = [[\theta]] = [[\theta \hat{n}]] \tag{6}$$

where

$$[[u]] \equiv \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix} \tag{7}$$

the equations in n dimensions assume their familiar three-dimensional forms,[4]

$$A = \exp\left([[\theta]]\right) \tag{8}$$

$$A = \left(I + [[g]]\right)\left(I - [[g]]\right)^{-1} \tag{9}$$

and

$$g = \tan(\theta/2)\hat{n} \tag{10}$$

(Note that we have chosen a different sign convention than Bar-Itzhack and Markley.)

Attitude Determination Problem

Suppose that we are given N linearly independent vector measurements W_k, $k = 1, \ldots, N$, which are the representations of N n-dimensional vectors in the spacecraft reference frame. These are related to V_k, $k = 1, \ldots, N$, the representations of these same vectors in the primary reference frame according to

$$W_k = A V_k, \qquad k = 1, \ldots, N \tag{11}$$

where A is assumed to be an $n \times n$ proper orthogonal matrix. Equation (11) assumes that the measurements are perfect, i.e., noise free, which should be no more true in practice in n dimensions than it is in three dimensions. (Fortunately, there is not a well-established practice for dimensions higher than three.) The assumption implied by Eq. (11), however, is similar to the one behind the triad method,[5,6] so that Eq. (11) is not without precedent.

The first question to be answered is, What is the minimum value of N that permits a unique solution? We remark first that the N vectors amount to (Nn) total components. At the same time, from Eq. (11), the N vectors are subject to $N(N+1)/2$ constraints of the form

$$W_i \cdot W_j = V_i \cdot V_j, \qquad i = 1, \ldots, N, \qquad j = 1, \ldots, N \tag{12}$$

Thus, the total number of unconstrained components in the vector measurements, i.e., the number of components that carry independent information about the attitude, is $Nn - N(N+1)/2$. The minimum number of measurements, then, is the minimum value of N satisfying

$$Nn - \frac{N(N+1)}{2} \geq \frac{n(n-1)}{2} \tag{13}$$

The equality, in fact, has two solutions, $n-1$ and n. Thus,

$$N_{\min} = n - 1 \tag{14}$$

Ebert[7] has offered a geometrical argument for this result. Suppose that we are given $n-m$ vector measurements. These $n-m$ vectors span a subspace $\mathcal{V}_{n-m}$ of dimension $n-m$ of the n-dimensional vector space $\mathcal{V}_n$. The vector space $\mathcal{V}_n$ may be written, therefore, as the direct sum $\mathcal{V}_n = \mathcal{V}_{n-m} \oplus \mathcal{V}_m$ of this subspace and the subspace of vectors perpendicular to the $n-m$ measurements. Clearly, any rotation within $\mathcal{V}_m$ will leave the $n-m$ measurements unchanged. Thus, for the $n-m$ measured vectors to define the attitude uniquely, it is necessary that the dimension of $\mathcal{V}_m$ be so small that rotations are not possible. Thus, we must have either $m = 0$ or 1. The latter value leads to the smaller value of $n-m$ and Eq. (14).

Thus, the attitude determination problem is to determine an $n \times n$ proper orthogonal matrix A given $n-1$ linearly independent n-dimensional vector measurements W_k, $k = 1, \ldots, n-1$.

General n-Dimensional Algorithm

Given the $n-1$ vectors V_k, $k = 1, \ldots, n-1$, we construct a set of $n-1$ orthonormal vectors $\hat{r}_k$, $k = 1, \ldots, n-1$, according to the Gram-Schmidt orthogonalization.[8] Thus,

$$r_1 = V_1, \qquad \hat{r}_1 = r_1/|r_1|$$

$$r_2 = V_2 - (\hat{r}_1 \cdot V_2)\hat{r}_1, \qquad \hat{r}_2 = r_2/|r_2|$$

$$\vdots \qquad\qquad \vdots$$

$$r_{n-1} = V_{n-1} - \sum_{k}^{n-2} (\hat{r}_k \cdot V_{n-1})\hat{r}_k, \qquad \hat{r}_{n-1} = r_{n-1}/|r_{n-1}| \tag{15}$$

Similarly, we construct $n-1$ orthonormal vectors $\hat{s}_k$, $k = 1, \ldots, n-1$, from the W_k, $k = 1, \ldots, n-1$. It follows from Eq. (11) that

$$\hat{s}_k = A\hat{r}_k \tag{16}$$

We now construct the vector $\hat{r}_n$ (and in a corresponding fashion $\hat{s}_n$) according to

$$(\hat{r}_n)_\ell = \sum_{i_1, i_2, \ldots, i_{n-1}} \epsilon_{i_1 i_2 \ldots i_{n-1}\ell} (\hat{r}_1)_{i_1} (\hat{r}_2)_{i_2} \cdots (\hat{r}_{n-1})_{i_{n-1}} \tag{17}$$

where $(\hat{r}_k)_m$ denotes the mth component of $\hat{r}_k$, and the sums over each index are from 1 to n. The quantity $\epsilon_{i_1 i_2 \ldots i_n}$ is the Levi-Civita symbol in n indices, which is just the parity of the permutation taking $(1, 2, \ldots, n)$ into $(i_1, i_2, \ldots, i_n)$ and vanishes if the latter is not a permutation of the former. Thus, the Levi-Civita symbol satisfies

$$\epsilon_{123\cdots n} = 1 \tag{18}$$

$$\epsilon_{i_1 \cdots i_j i_{j+1} \cdots i_n} = -\epsilon_{i_1 \cdots i_{j+1} i_j \cdots i_n} \tag{19}$$

The determinant of an $n \times n$ matrix M is often defined in terms of the Levi-Civita symbol[9] as

$$\det M \equiv \sum_{i_1, i_2, \ldots, i_n} \epsilon_{i_1 i_2 \cdots i_n} M_{1 i_1} M_{2 i_2} \cdots M_{n i_n} \tag{20}$$

where $M_{k i_k}$ denotes the (k, i_k) element of M. From Eq. (17)

$$\begin{aligned}(\hat{s}_n)_\ell &= \sum_{i_1, i_2, \ldots, i_{n-1}} \epsilon_{i_1 i_2 \cdots i_{n-1}\ell} (A\hat{r}_1)_{i_1} (A\hat{r}_2)_{i_2} \cdots (A\hat{r}_{n-1})_{i_{n-1}} \\ &= \sum_{i_1, i_2, \ldots, i_{n-1}} \epsilon_{i_1 i_2 \cdots i_{n-1}\ell} \sum_{j_1, j_2, \ldots, j_{n-1}} A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_{n-1} j_{n-1}} \\ &\quad \times (\hat{r}_1)_{j_1} (\hat{r}_2)_{j_2} \cdots (\hat{r}_{n-1})_{j_{n-1}}\end{aligned} \tag{21}$$

Thus,

$$\begin{aligned}(A^T\hat{s}_n)_\ell &= \sum_{i_n} (A^T)_{\ell i_n} (\hat{s}_n)_{i_n} = \sum_{i_n} A_{i_n \ell} (\hat{s}_n)_{i_n} = \sum_{i_1, i_2, \ldots, i_n} \epsilon_{i_1 i_2 \cdots i_n} \\ &\quad \times \sum_{j_1, j_2, \ldots, j_{n-1}} A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_{n-1} j_{n-1}} A_{i_n \ell} \\ &\quad \times (\hat{r}_1)_{j_1} (\hat{r}_2)_{j_2} \cdots (\hat{r}_{n-1})_{j_{n-1}} = (\det A) \sum_{j_1, j_2, \ldots, j_{n-1}} \epsilon_{j_1 j_2 \cdots j_{n-1}\ell} \\ &\quad \times (\hat{r}_1)_{j_1} (\hat{r}_2)_{j_2} \cdots (\hat{r}_{n-1})_{j_{n-1}} = (\hat{r}_n)_\ell\end{aligned} \tag{22}$$

so that

$$\hat{s}_n = A\hat{r}_n \tag{23}$$

We can show likewise from the properties of the Levi-Civita symbol in n dimensions that

$$\hat{r}_k \cdot \hat{r}_n = 0, \qquad k = 1,\dots,n-1 \tag{24a}$$

$$\hat{s}_k \cdot \hat{s}_n = 0, \qquad k = 1,\dots,n-1 \tag{24b}$$

$$\hat{r}_n \cdot \hat{r}_n = \hat{s}_n \cdot \hat{s}_n = 1 \tag{25}$$

It follows that the two $n \times n$ matrices R and S defined according to their columns by

$$R = [\hat{r}_1, \hat{r}_2, \cdots, \hat{r}_n], \qquad S = [\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n] \tag{26}$$

are each proper orthogonal and

$$S = AR \tag{27}$$

Thus,

$$A = SR^T \tag{28}$$

is proper orthogonal and is the desired $n \times n$ attitude matrix.

Comparison with the Triad Method

This method bears a clear resemblance to the triad method[5,6] first published by Black[5] in 1964 to solve this particular attitude estimation problem in three dimensions. The resemblance is made all the stronger if we note that Eq. (17) can be written as

$$(\hat{r}_n)_\ell = \det|\hat{r}_1\ \hat{r}_2\ \cdots\ \hat{r}_{n-1}\ \mathbf{1}_\ell| \tag{29}$$

where $\mathbf{1}_\ell$ denotes a column vector every element of which vanishes except the ℓth, which is unity. Equation (29) is the generalization in n dimensions of the vector product, which is prominent in the triad method. The triad algorithm, in fact, had already been in use in the previous decade,[10] and Eq. (28) as the relation between a rotation matrix and the two orthonormal bases it connects can be found for three dimensions in dyadic form in the works of Gibbs.[11] One can only imagine that if Eq. (11) were set before Gibbs he would have computed the attitude matrix via the triad method.

The triad method has been variously called the algebraic method[12] or the Sun-Mag method, after the once two most commonly used sensors to which this method was applied. In the triad method, the two sets of orthonormal matrices are computed according to

$$\hat{r}_1' = V_1/|V_1| \tag{30a}$$

$$\hat{r}_2' = V_1 \times V_2/|V_1 \times V_2| \tag{30b}$$

$$\hat{r}_3' = \hat{r}_1' \times \hat{r}_2' \tag{30c}$$

and correspondingly for $\hat{s}_k'$, $k = 1,2,3$. The primes distinguish the quantities in the triad method from those just defined.

Clearly, $\hat{r}_1'$ is identical to $\hat{r}_1$. From the well-known Grassman identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} \tag{31}$$

we see that the definition of $\hat{r}_3'$ is equivalent to

$$r_3' = -V_2 + (\hat{r}_1 \cdot V_2)\hat{r}_1' \tag{32a}$$

$$\hat{r}_3' = r_3'/|r_3'| \tag{32b}$$

Thus,

$$\hat{r}_3' = -\hat{r}_2 \tag{33}$$

Finally,

$$(\hat{r}_2')_k = \sum_{i,j} \epsilon_{ijk}(\hat{r}_1)_i(\hat{r}_2)_j \tag{34}$$

so that

$$\hat{r}_2' = \hat{r}_3 \tag{35}$$

Thus, in the triad method

$$R' = [\hat{r}_1', \hat{r}_2', \hat{r}_3'] = [\hat{r}_1, -\hat{r}_3, \hat{r}_2] \tag{36a}$$

$$S' = [\hat{s}_1', \hat{s}_2', \hat{s}_3'] = [\hat{s}_1, -\hat{s}_3, \hat{s}_2] \tag{36b}$$

so that

$$R' = R\Theta, \qquad S' = S\Theta \tag{37}$$

where

$$\Theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{38}$$

is evidently proper orthogonal. The triad solution for the attitude matrix is thus,

$$A = S'R'^T = SR^T \tag{39}$$

and is, therefore, a special case of the general n-dimensional method.

Conclusions

The problem of estimating an $n \times n$ proper orthogonal matrix from N linearly independent vector measurements has been considered. It has been shown that a unique solution for the proper orthogonal matrix exists provided that $N = n - 1$. An algorithm for constructing the attitude matrix based on the Gram-Schmidt orthogonalization method was presented. The special case of this algorithm for $n = 3$ was shown to be equivalent to the well-known triad algorithm.

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